MS221/W

The Open University

Course Examination 2005

Exploring Mathematics

Friday 21 October 2005 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can. Credit will be given for answers to no more than TWO questions in Part II. Your answers to each part should be written in the answer books provided.

In most questions, some marks will be awarded for intermediate steps in the working. A correct answer, not supported by working – as, for example, one taken from a calculator program – may not receive full credit.

72% of the available marks are assigned to Part I and 28% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

You are advised not to cross through any work until you have replaced it with another solution to the same question. In Part II of the paper, if you submit attempts to more than two questions, your best two scores will count towards your result.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put all your used answer books and your question paper together, with your signed desk record on top. Fix them all together with the fastener provided.

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PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 72% of the available examination marks. The allocation of marks for each question is shown beside the question number.*

(iii) *You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.*

Question 1 – 4 marks

(a) Express the translation

$$(x, y) \longmapsto (x + 7, y - 3)$$

in the form $t_{p,q}$. [1]

(b) Write down the translation that moves the origin to the point $(-5, 2)$ in the form

$$(x, y) \longmapsto (\quad , \quad).$$ [1]

(c) Write down the composition of the two translations from parts (a) and (b) in the form

$$t_{p,q}\colon (x, y) \longmapsto (\quad , \quad).$$ [2]

Question 2 – 6 marks

A conic in standard position has the following parametrisation:

$$x = 5t^2, \quad y = 10t.$$

(a) Find the equation of the conic, and state what type of conic it is. [1]

(b) Write down

(i) the eccentricity of the conic;

(ii) the coordinates of any foci;

(iii) the equations of any directrices. [3]

(c) Sketch the conic, marking on your sketch the features from part (b). [2]

Question 3 – 8 marks

This question is about the recurrence system

$$u_0 = 8, \quad u_1 = 1, \quad u_{n+2} = \tfrac{3}{5}u_{n+1} + \tfrac{2}{5}u_n \quad (n = 0, 1, 2, \ldots).$$

(a) Find u_2. [1]

(b) Find a closed form for the sequence u_n. [5]

(c) Explain what happens to u_n in the long term, that is, for large n. [2]

Justify your answer briefly.

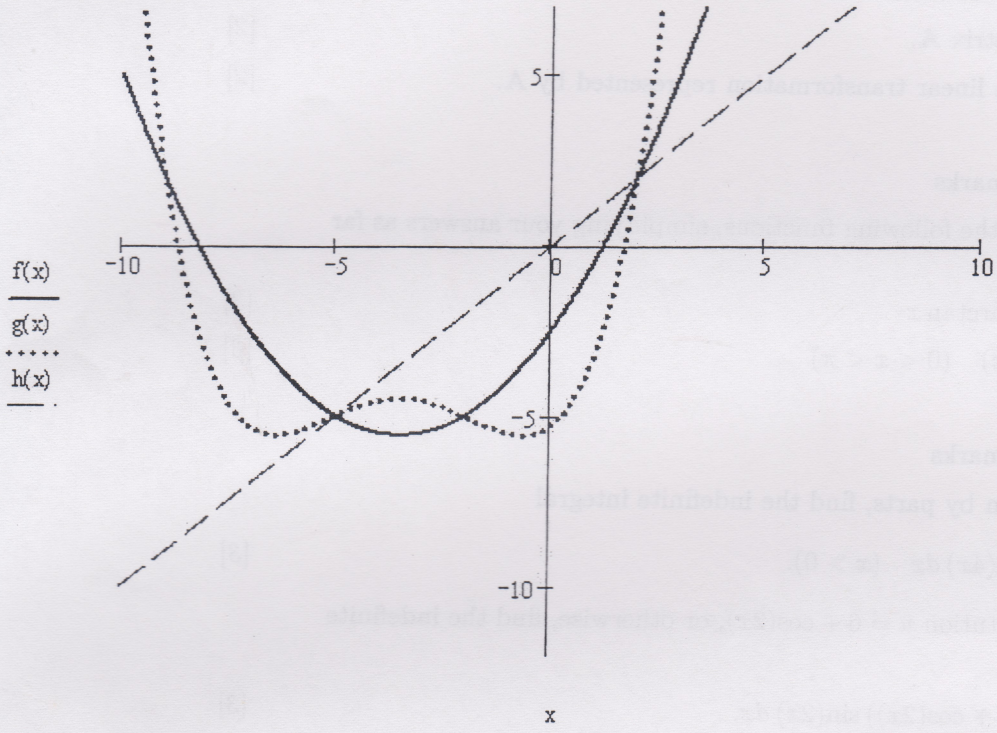
Question 4 – 7 marks

This question is about the function $f(x) = \frac{1}{4}x^2 + \frac{7}{4}x - \frac{5}{2}$.

(a) Find algebraically the fixed points of f. [2]

(b) Classify the fixed points of f as attracting, repelling or indifferent, explaining your reasoning clearly. [3]

(c) The following Mathcad diagram shows the graphs of f and of the composite function $f \circ f$, together with the line $y = x$. Use the diagram to decide whether f has a 2-cycle, explaining your reasoning clearly.



[2]

Question 5 – 3 marks

The matrix $\mathbf{A}$ can be expressed as $\mathbf{A} = \mathbf{PDP}^{-1}$, where

$$\mathbf{P} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}.$$

Use this fact to calculate $\mathbf{A}^5$ without calculating $\mathbf{A}$, showing your working. [3]

Question 6 – 8 marks

In this question, f is the linear transformation represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}.$$

(a) The unit square has vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.

(i) Find the image under f of each of the vertices of the unit square. [1]

(ii) Plot the unit square and its image under f on the same axes. [2]

(b) The matrix of f can be expressed as $\mathbf{M} = \mathbf{BA}$, where $\mathbf{B}$ represents a rotation through $\pi/2$ anticlockwise about the origin.

(i) Write down the matrix $\mathbf{B}$. [1]

(ii) Find the matrix $\mathbf{A}$. [2]

(iii) Describe the linear transformation represented by $\mathbf{A}$. [2]

Question 7 – 6 marks

Differentiate each of the following functions, simplifying your answers as far as possible.

(a) $f(x) = (1 + x^2) \arctan x$ [3]

(b) $g(x) = \ln(\operatorname{cosec} x) \quad (0 < x < \pi)$ [3]

Question 8 – 6 marks

(a) Using integration by parts, find the indefinite integral

$$\int x^{3/2} \ln(4x)\, dx \quad (x > 0).$$ [3]

(b) Using the substitution $u = 6 + \cos(2x)$, or otherwise, find the indefinite integral

$$\int \exp\left(6 + \cos(2x)\right) \sin(2x)\, dx.$$ [3]

Question 9 – 6 marks

(a) Using standard Taylor series, find the Taylor series about 0 for each of the following functions, giving in each case the first three non-zero terms.

(i) $f(x) = x \cos x$ [1]

(ii) $g(x) = \sin x - x \cos x$ [2]

(b) By differentiating the series for $g(x)$ term by term, find the Taylor series about 0 for $g'(x)$, giving the first three non-zero terms and simplifying the coefficients. [2]

(c) What function, involving $\sin x$, has a Taylor series about 0 whose first three non-zero terms are the same as those for $g'(x)$? [1]

Question 10 – 6 marks

Let w be the complex number $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

(a) Find the modulus and argument of the complex number w. [2]

(b) Hence write down w in polar form. [1]

(c) Show that $\overline{w} = w^2$. [1]

(d) Explain why $w^3 = 1$. [1]

(e) For $z = \langle r, \theta \rangle$, find the product wz in polar form. [1]

Question 11 – 6 marks

Use Euclid's Algorithm to find the multiplicative inverse of 18 in $\mathbb{Z}_{43}$. [6]

Question 12 – 6 marks

The variable propositions $a(n)$, $b(n)$, $c(n)$ and $d(n)$, where $n \in \mathbb{N}$, have meanings given below.

$a(n)$ means: n is divisible by 6

$b(n)$ means: n is divisible by 12

$c(n)$ means: n is divisible by 15

$d(n)$ means: n is divisible by 24

(a) Using a combination of two of the propositions above, give a condition that is necessary and sufficient for n to be divisible by 120. [2]

(b) Consider the proposition (A) given below.

(A) $a(n) \wedge c(n) \Rightarrow d(n)$

(i) Give an example of a number for which proposition (A) is false. [1]

(ii) Write down the converse of proposition (A), both in words and in symbols. [3]

PART II

Instructions

(i) *Credit will be given for answers to no more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 14% of the total marks for the examination.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 13

In this question you are expected to show all of your working, using exact values in surd form. Approximate numerical answers (other than the value of θ) taken from a calculator will not receive any credit.

In this question L is the conic with equation

$$30x^2 - 14\sqrt{2}xy + 37y^2 - 1012 = 0.$$

(a) Use the result in the Handbook to show that if θ is the inclination of L, then $\tan 2\theta = 2\sqrt{2}$. [1]

(b) Use the *exact* value of $\tan(2\theta)$ from part (a) and the formula

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta},$$

to show that the exact value of $\tan\theta$ is $\frac{1}{\sqrt{2}}$. [2]

(c) Use a suitable triangle to find the exact values of $\sin\theta$ and $\cos\theta$. [2]

(d) Hence find the exact equation of the conic K, in standard position or reflected standard position, which is such that $L = r_\theta(K)$. [2]

(e) Find the inclination θ of L in degrees to 2 decimal places, and hence draw a sketch of the conic L, with its axes. [3]

(f) Find the exact value of the eccentricity of K. [1]

(g) State, with a brief reason, the eccentricity of L. [1]

(h) Find the equations of the two axes of L. [2]

Question 14

(a) In this part, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 6 & -3 \end{pmatrix}.$$

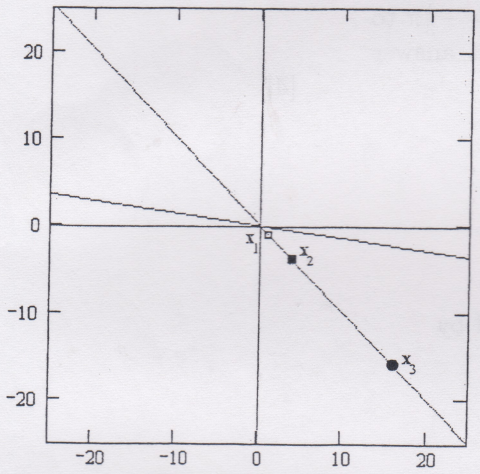
(i) Find the eigenvalues of $\mathbf{A}$. [2]

(ii) Find the eigenlines of $\mathbf{A}$ and, for each eigenline, write down a corresponding eigenvector with integer components. [4]

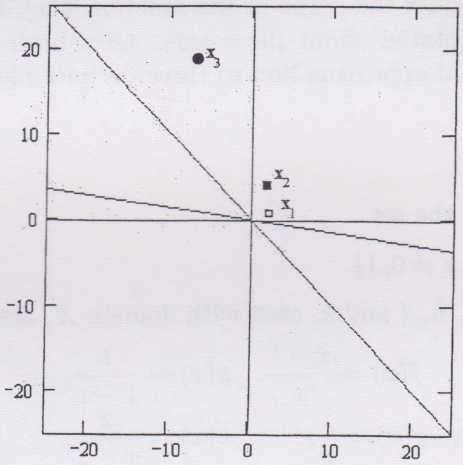
(iii) Write down an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. (You do not need to evaluate $\mathbf{P}^{-1}$.) [2]

(b) In this part, $\mathbf{B}$ is a matrix with eigenvalues 2 and -4, and corresponding eigenlines $y = -\frac{1}{7}x$ and $y = -x$, respectively. Each of diagrams (i) to (iii) below shows a Mathcad plot of the first three points of an iteration sequence; the lines drawn are the axes and the eigenlines of $\mathbf{B}$. In each case, explain why the iteration sequence *cannot* have recurrence relation $\mathbf{x}_{n+1} = \mathbf{B}\mathbf{x}_n$. [6]

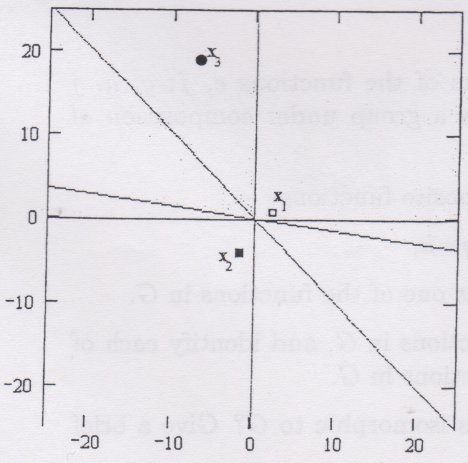
(i)



(ii)



(iii)



Question 15

This question concerns the function

$$f(x) = \frac{1+\sin x}{\cos x} \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi).$$

(a) (i) Show that $f(x) > 0$ throughout the domain of the function f. [1]

(ii) Show that f is an increasing function. [2]

(b) By using the substitution $u = 1 - \sin x$, show that

$$\int f(x)\,dx = -\ln(1-\sin x) + c,$$

where c is an arbitrary constant. [4]

(c) Hence find the area under the graph of $f(x)$ between $x = -\frac{1}{3}\pi$ and $x = \frac{1}{3}\pi$, giving both an exact answer and a decimal approximation to three decimal places. [2]

(d) (i) Show that the rule for the function f can also be written as

$$f(x) = \sec x + \tan x. \quad [1]$$

(ii) Hence, or otherwise, find the volume of revolution obtained when the region under the graph of the function $f(x)$, from $x = -\frac{1}{3}\pi$ to $x = \frac{1}{3}\pi$, is rotated about the x-axis. Give both an exact answer and a decimal approximation to three decimal places. [4]

Question 16

In this question X is the set

$$X = \{x \in \mathbb{R} : x \neq 0, 1\}.$$

The functions e, f, g, h, j and k, each with domain X, are defined by

$$e(x) = x, \qquad f(x) = \frac{x-1}{x}, \quad g(x) = \frac{1}{1-x},$$
$$h(x) = 1-x, \quad j(x) = \frac{1}{x}, \qquad k(x) = \frac{x}{x-1}.$$

(a) Find the image set of f. [2]

(b) Show that the composite function $f \circ f$ can be formed, and find the rule for $f \circ f$. [3]

G is the set of functions

$$G = \{e, f, g, h, j, k\}.$$

You may assume that all compositions of the functions e, f, g, h, j and k are possible, and that G forms a group under composition of functions.

(c) Find the rule for each of the four composite functions

$$f \circ g, \quad k \circ k, \quad k \circ g \quad \text{and} \quad g \circ k,$$

and identify each composite function as one of the functions in G. [4]

(d) Find the inverse of each of the six functions in G, and identify each of the inverse functions as one of the functions in G. [3]

(e) Which of the two groups $S(\Delta)$ or $\mathbb{Z}_6$ is isomorphic to G? Give a brief reason for you choice. [2]

[END OF QUESTION PAPER]